

MAS317 Linear Algebra II, 2007

Solutions to Coursework 9.

1. a). $U + W \supseteq U$ so $(U + W)^\perp \subseteq U^\perp$ and $U + W \supseteq W$ so $(U + W)^\perp \subseteq W^\perp$; so $(U + W)^\perp \subseteq U^\perp \cap W^\perp$. [4 marks]

Now if $x \in U^\perp \cap W^\perp$ and $y \in U + W$, then $y = u + w$ with $u \in U$ and $w \in W$, then $f(x, u) = 0$ as $x \in U^\perp$ and $f(x, w) = 0$ as $x \in W^\perp$, and so $f(x, y) = f(x, u + w) = f(x, u) + f(x, w) = 0 + 0 = 0$. Hence $x \in (U + W)^\perp$. So $U^\perp \cap W^\perp \subseteq (U + W)^\perp$, and so they are equal.

[6 marks]

- b). $(U \cap W) \subseteq U$ and so $(U \cap W)^\perp \supseteq U^\perp$ and $(U \cap W) \subseteq W$ and so $(U \cap W)^\perp \supseteq W^\perp$, and so $(U \cap W)^\perp \supseteq U^\perp + W^\perp$. [5 marks]

- c). Given f on $\mathbb{R}^2$ by $f((x_1, x_2), (y_1, y_2)) = x_1 y_1$.

Let $U = \langle (1, 0) \rangle$ and $W = \langle (1, 1) \rangle$.

One example: Then $U \cap W = \{0\}$ and so $(U \cap W)^\perp = \mathbb{R}^2$,

But $(x_1, x_2) \in U^\perp$ if $f((x_1, x_2), (1, 0)) = 0$ and so if $x_1 = 0$, and $(x_1, x_2) \in W^\perp$ if $f((x_1, x_2), (1, 1)) = 0$ and so if $x_1 = 0$. [10 marks]

So $U^\perp = W^\perp = \langle (0, 1) \rangle$, and so $U^\perp + W^\perp = \langle (0, 1) \rangle \neq (U \cap W)^\perp$.

2. $f((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)) = x_1 y_2 + x_2 y_3 + x_3 y_4$.

- (i) Let $u = (x_1, x_2, x_3, x_4) \in F^4$, then $u \in V^\perp$ iff $u \perp \{e_1, e_2, e_3, e_4\}$ iff $f(u, e_i) = 0$ for $i = 1, 2, 3, 4$. iff $0 = 0, x_1 = 0, x_2 = 0$ and $x_3 = 0$. So $V^\perp = \{(0, 0, 0, x_4) \mid x_4 \in F\}$. [5 marks]

- (ii) Let $u = (x_1, x_2, x_3, x_4) \in F^4$, then $u \in U^\perp$ iff $u \perp \{(1, 0, 0, 0), (0, 0, 1, 1)\}$ iff $0 = 0$ and $x_2 + x_3 = 0$. So $U^\perp = \{(x_1, x_2, -x_2, x_4) \mid x_1, x_2, x_4 \in F\}$. [5 marks]

$U^\perp$ has basis $e_1, e_2 - e_3, e_4$, so if $u = (x_1, x_2, x_3, x_4) \in F^4$, then $u \in (U^\perp)^\perp$ iff $u \perp \{e_1, e_2 - e_3, e_4\}$ iff $0 = 0$ and $x_1 - x_2 = 0$ and $x_3 = 0$.

So $(U^\perp)^\perp = \{(x_1, x_1, 0, x_4) \mid x_1, x_4 \in F\}$. [5 marks]

$(U^\perp)^\perp$ has basis $e_1 + e_2, e_4$, so if $u = (x_1, x_2, x_3, x_4) \in F^4$, then

$u \in ((U^\perp)^\perp)^\perp$ iff $u \perp \{e_1 + e_2, e_4\}$ iff $x_1 = 0$ and $x_3 = 0$.

So $((U^\perp)^\perp)^\perp = \{(0, x_2, 0, x_4) \mid x_2, x_4 \in F\}$. [5 marks]

$((U^\perp)^\perp)^\perp$ has basis e_2, e_4 , so if $u = (x_1, x_2, x_3, x_4) \in F^4$, then

$u \in (((U^\perp)^\perp)^\perp)^\perp$ iff $u \perp \{e_2, e_4\}$ iff $x_1 = 0$ and $x_3 = 0$.

So $((((U^\perp)^\perp)^\perp)^\perp)^\perp = ((U^\perp)^\perp)^\perp$.

So the sequence of subspaces $U^\perp, (U^\perp)^\perp, ((U^\perp)^\perp)^\perp, \dots$ is:

$\langle e_1, e_2 - e_3, e_4 \rangle, \langle e_1 + e_2, e_4 \rangle, \langle e_2, e_4 \rangle, \langle e_2, e_4 \rangle, \langle e_2, e_4 \rangle, \dots$ [5 marks]

3. (i) $A = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 3 & -1 \\ -2 & -1 & -5 \end{pmatrix}$, and define the bilinear form f on $\mathbb{R}^3$ by

$f(u, v) = uAv^T$ for all $u, v \in \mathbb{R}^3$.

f is symmetric as A is symmetric so use (8.9). Normal form is a diagonal matrix N with diagonal entries each 1, -1 , or 0 as this is over $\mathbb{R}$. [5 marks]

First find v_1 s.t. $f(v_1, v_1) \neq 0$. *There is lots of choice here; eg could take $v_1 = e_1 = (1, 0, 0)$ as $f(e_1, e_1) = (A)_{11} = 1$.* [5 marks]

Now v_2 and v_3 must be orthogonal to e_1 so find $\langle e_1 \rangle^\perp$. [Note: As A is symmetric, so you can do the calculation with e_1 on the right or the left.]

$(x_1, x_2, x_3) \perp e_1$ iff $(x_1, x_2, x_3)Ae_1 = 0$ iff $x_1 + 2x_2 - 2x_3 = 0$.

So v_2 must be a solution to this and have $f(v_2, v_2) \neq 0$ if this is possible; do this by trial and error, choose a vector and find $f(v_2, v_2)$. *There is lots of choice here; eg could take $v_2 = (-2, 1, 0)$, then $f(v_2, v_2) = (-2, 1, 0)A(-2, 1, 0) = (-2, 1, 0)(0, -1, 3)^T = -1$.*

[5 marks]

Now v_3 must be orthogonal to e_1 and to $(-2, 1, 0)$, so if $v_3 = (x_1, x_2, x_3)$ then $x_1 + 2x_2 - 2x_3 = 0$ and $(x_1, x_2, x_3)A(-2, 1, 0) = 0$ so $x_1 + 2x_2 - 2x_3 = 0$ and $-x_2 + 3x_3 = 0$. So put $u = (-4, 3, 1)$. Now $f(u, u) = (-4, 3, 1)A(-4, 3, 1) = (-4, 3, 1)(0, 0, 0)^T = 0$. [5 marks]

Now v_1, v_2, v_3 are an orthogonal basis for $\mathbb{R}^3$, as these vectors are orthogonal to each other. and so the normal form for A is [2 marks]

$N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. P has the basis vectors as its columns so

$$P = \begin{pmatrix} 1 & -2 & -4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}. \quad [3 \text{ marks}]$$

(ii) $A = \begin{pmatrix} 0 & 3 & 6 & 3 \\ -3 & 0 & 1 & 3 \\ -6 & -1 & 0 & 0 \\ -3 & -3 & 0 & 0 \end{pmatrix}$, and define the bilinear form f on $\mathbb{R}^4$ by

$$f(u, v) = uAv^T \text{ for all } u, v \in \mathbb{R}^4.$$

f is alternating as A is alternating (and normal form is $Alt(4, 2)$ as A clearly has row rank > 2). We use (8.11).

We want u_1, u_2 such that $f(u_1, u_2) \neq 0$. *There is lots of choice here; eg* $u_1 = e_1$ and $u_2 = e_2$ as $f(e_1, e_2) = (A)_{12} = 3$.

Put $v_1 = (1/3)e_1$ and $v_2 = e_2$ and the matrix of the restriction of f to $\langle v_1, v_2 \rangle$ has matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. [8 marks]

Now v_3 and v_4 must be orthogonal to $(1/3)e_1$ and to e_2 .

(x_1, x_2, x_3, x_4) is orthogonal to $(1/3)e_1$ and to e_2 . iff
 $(x_1, x_2, x_3, x_4)A(1/3)e_1 = 0$ and $(x_1, x_2, x_3, x_4)Ae_2 = 0$. iff
 $x_2 + 2x_3 + x_4 = 0$ and $-3x_1 + x_3 + 3x_4 = 0$.

We need two linearly independent solutions for v_3 and v_4 . *There is lots of choice here; eg* let $u_3 = (1, -6, 3, 0)$ and $u_4 = (1, -1, 0, 1)$. Then
 $f(u_3, u_4) = (1, -6, 3, 0)A(1, -1, 0, 1)^T = (1, -6, 3, 0)(0, 0, -5, 0)^T = -15$.

So put $v_3 = (-1/15)u_3$ and $v_4 = u_4$ (*need to multiply one of them by $(-1/15)$*). [12 marks]

Then $N = Alt(4, 2)$ and

P has the basis vectors $v_1, \dots, v_4$ as its columns so

$$P = \begin{pmatrix} 1/3 & 0 & -1/15 & 1 \\ 0 & 1 & 2/5 & -1 \\ 0 & 0 & -1/5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad [5 \text{ marks}]$$